JACOB'S LADDERS, NEW PROPERTIES OF THE FUNCTION $\arg\zeta\left(\frac{1}{2}+it\right)$ AND CORRESPONDING METAMORPHOSES

JAN MOSER

Abstract. The notion of the Jacob's ladders, reversely iterated integrals and the ζ-factorization is used in this paper in order to obtain new results in study of the function $\arg\zeta\left(\frac{1}{2}+it\right)$. Namely, we obtain new formulae for non-local and non-linear interaction of the functions $|\zeta\left(\frac{1}{2}+it\right)|$ and $\arg\zeta\left(\frac{1}{2}+it\right)$, and also a set of metamorphoses of the oscillating Q-system.

1. Introduction

1.1. Let us denote by $N(T)$ the number of zeroes $\beta+i\gamma$ of the $\zeta(s)$-function such that

$$\beta\in(0,1),\ \gamma\in(0,T).$$

We suppose that T is not equal to any γ. Otherwise, we put

$$N(T)=\frac{1}{2}\lim_{\epsilon\to 0^+}[N(T+\epsilon)+N(T-\epsilon)].$$

It us well-know that

$$N(T)=\frac{1}{2\pi}T\ln\frac{T}{2\pi e}+\frac{7}{8}+S(T)+\mathcal{O}\left(\frac{1}{T}\right),$$

where

$$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right), \tag{1.1}$$

and the value of arg is obtained by continuous variation along the straight lines joining the points

$$2,\ 2+iT,\ \frac{1}{2}+iT,$$

starting with the value zero. Next, we have the function

$$S_1(T)=\int_0^T S(t)\mathrm{d}t. \tag{1.2}$$

1.2. Further, let us remind the following facts

$$\zeta\left(\frac{1}{2}+it\right)=\left|\zeta\left(\frac{1}{2}+it\right)\right|e^{i\arg\zeta\left(\frac{1}{2}+it\right)},$$

i.e. the functions

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|,\ \arg\zeta\left(\frac{1}{2}+it\right) \tag{1.3}$$

Key words and phrases. Riemann zeta-function.

are parts of the Riemann function

$$\zeta\left(\frac{1}{2}+it\right).$$

The study of these functions have proceeded by isolated ways. Namely:

(a) the first one studied by Hardy-Littlewood

$$\int_T^{T+U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \sim U\ln T, \dots$$

(b) the second one by Backlund, E. Landau, H. Bohr, Littlewood, Titchmarsh - to the fundamental Selberg's results (see [3], [4]).

Let us mention two of the Selberg's results:

$$\begin{gathered}\int_T^{T+H}\{S_1(t)\}^{2l}\mathrm{d}t = c_l H + \mathcal{O}\left(\frac{H}{\ln T}\right), \\ T^a \leq H \leq T;\ \frac{1}{2} < a \leq 1,\ l\in\mathbb{N},\end{gathered} \tag{1.4}$$

where l is arbitrary and fixed, (see [4], p. 130), and

$$S_1(t) = \Omega_{\pm}\left\{(\ln t)^{1/3}(\ln\ln t)^{-10/3}\right\}, \tag{1.5}$$

(see [4], p.150).

Remark 1. For our purpose it is sufficient to use the formula (1.4) in the minimal case

$$H = T^{1/2+\epsilon},\ \epsilon > 0,$$

where ϵ is sufficiently small (non-principal improvements of the exponent 1/2 are not relevant for our purpose).

1.3. To this date, there is no result in the theory of the Riemann zeta-function about the interaction of the functions (1.3), or the functions

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|, S_1(t). \tag{1.6}$$

That is, there is nothing known like

$$F\left(\left|\zeta\left(\frac{1}{2}+it\right)\right|, |S_1(\tau)|\right) = 0$$

for a set of values t, τ.

On the other hand, we have developed (see [1]) the method of ζ-factorization that gives, for example, the following formula (see [1], (1.7))

$$\frac{1}{\sqrt{|\zeta\left(\frac{1}{2}+i\alpha_0\right)|}} \sim \frac{1}{\sqrt{\Lambda}}\prod_{r=1}^k\left|\zeta\left(\frac{1}{2}+i\alpha_r\right)\right|$$

together with the infinite set of corresponding metamorphoses of the main multiform.

In this paper we use this method to obtain a result of the new type

$$|S_1(\alpha_0)| \sim \Phi\left\{\prod_{r=1}^k\left|\frac{\zeta\left(\frac{1}{2}+i\alpha_r\right)}{\zeta\left(\frac{1}{2}+i\beta_r\right)}\right|\right\} \tag{1.7}$$

together with the infinite set of metamorphoses of the corresponding Q-system from [2].

Remark 2. A kind of nonlinear and nonlocal interaction of the functions (1.6) is expressed by the formula (1.7).

2. Theorem

2.1. We begin with the Selberg's formula

$$\begin{aligned} &\int_T^{T+H} \{S_1(t)\}^{2l}\mathrm{d}t \sim c_l H,\ T\to\infty, \\ &H = T^{1/2+\epsilon},\ l\in\mathbb{N},\ \epsilon>0, \end{aligned} \tag{2.1}$$

(comp. (1.4) and Remark 1), where l is arbitrary and fixed, ϵ is sufficiently small. Now, if we use our method of transformation (see [2], (4.1)–(4.19)) in the case of the formula (2.1) then we obtain (see (1.1), (1.2)) the following

Theorem. Let

$$[T,T+H] \longrightarrow [\overset{1}{T},\overset{1}{\widehat{T+H}}],\dots,[\overset{k}{T},\overset{k}{\widehat{T+H}}], \tag{2.2}$$

where

$$[\overset{r}{T},\overset{r}{\widehat{T+H}}],\ r=1,\dots,k,\ k\le k_0,\ k_0\in\mathbb{N}$$

be the reversely iterated segment corresponding to the first segment in (2.2) and k_0 be an arbitrary and fixed number. Then there is a sufficiently big

$$T_0 = T_0(l,\epsilon) > 0$$

such that for every $T>T_0$ and every admissible l,ϵ,k there are the functions

$$\begin{aligned} &\alpha_r = \alpha_r(T,l;\epsilon,k),\ r=0,1,\dots,k, \\ &\beta_r = \beta_r(T;\epsilon,k),\ r=1,\dots,k, \\ &\alpha_r,\beta_r \neq \gamma:\ \zeta\left(\frac{1}{2}+i\gamma\right)=0 \end{aligned} \tag{2.3}$$

such that

$$\begin{aligned} &\left| \int_0^{\alpha_0(T)} \arg\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t \right| \sim \\ &\sim \pi(c_l)^{\frac{1}{2l}} \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2}+i\alpha_r(T,l)\right)}{\zeta\left(\frac{1}{2}+i\beta_r(T)\right)} \right|^{-\frac{1}{l}},\ T\to\infty. \end{aligned} \tag{2.4}$$

Moreover, the sequences

$$\{\alpha_r\}_{r=0}^k,\ \{\beta_r\}_{r=1}^k$$

have the following properties

$$\begin{aligned} &T<\alpha_0<\alpha_1<\dots<\alpha_k, \\ &T<\beta_1<\beta_2<\dots<\beta_k, \\ &\alpha_0\in(T,T+H),\ \alpha_r,\beta_r\in(\overset{r}{T},\overset{r}{\widehat{T+H}}), \\ &r=1,\dots,k, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\alpha_{r+1}-\alpha_r\sim(1-c)\pi(T),\ r=0,1,\dots,k-1, \\ &\beta_{r+1}-\beta_r\sim(1-c)\pi(T),\ r=1,\dots,k-1, \end{aligned} \tag{2.6}$$

where

$$\pi(T)\sim\frac{T}{\ln T},\ T\to\infty$$

is the prime-counting function and c is the Euler's constant.

Remark 3. Let us notice that the asymptotic behavior of the sets

$$\{\alpha_r\}_{r=0}^k,\ \{\beta_r\}_{r=1}^k \tag{2.7}$$

is as follows: at $T\to\infty$ the points of every set in (2.7) recede unboundedly each from other and all together recede to infinity. Hence, at $T\to\infty$ each set in (2.7) looks like one-dimensional Friedmann-Hubble universe.

2.2. Let us denote the mean-value of the function

$$\arg\zeta\left(\frac{1}{2}+it\right),\ t\in[0,T]$$

by the symbol

$$\langle\arg\zeta\left(\frac{1}{2}+it\right)\rangle\Big|_{[0,T]}.$$

Let us mention that the function under consideration has an infinite set of first-order discontinuities. Since

$$\int_0^{\alpha_0(T)}\arg\zeta\left(\frac{1}{2}+it\right){\rm d}t=\alpha_0(T)\ \langle\arg\zeta\left(\frac{1}{2}+it\right)\rangle\Big|_{[0,\alpha_0(T)]},$$

then we obtain from (2.4) the following

Corollary 1.

$$\begin{aligned} &\left|\langle\arg\zeta\left(\frac{1}{2}+it\right)\rangle\Big|_{[0,\alpha_0(T)]}\right|\sim \\ &\sim\frac{\pi(c_l)^{\frac{1}{2l}}}{\alpha_0(T)}\prod_{r=1}^k\left|\frac{\zeta\left(\frac{1}{2}+i\alpha_r(T,l)\right)}{\zeta\left(\frac{1}{2}+i\beta_r(T,l)\right)}\right|^{-\frac{1}{l}}, \\ &\alpha_0(T)\in(T,T+H),\ T\to\infty. \end{aligned} \tag{2.8}$$

Let us remind that the following Littlewood's estimate (comp. [5], p. 189)

$$S_1(t)=\mathcal{O}(\ln t),\ t\to\infty$$

holds true. Hence, we have (see (1.1), (1.2)) the estimate

$$\langle\arg\zeta\left(\frac{1}{2}+it\right)\rangle\Big|_{[0,T]}=\mathcal{O}\left(\frac{\ln T}{T}\right),\ T\to\infty. \tag{2.9}$$

Remark 4. Consequently, we have obtained in the direction of the estimate (2.9) the explicit asymptotic formula (2.8) for the mean-value

$$\langle\arg\zeta\left(\frac{1}{2}+it\right)\rangle\Big|_{[0,T]}$$

on the infinite subset

$$\{\alpha_0(T)\},\alpha_0(T)\in(T,T+T^{1/2+\epsilon}),\ T\to\infty.$$

3. Reduction of the integral in (2.4)

3.1. Now, we use the Selberg's Ω-theorem (1.5) to transform our formula (2.4). It follows from (1.5) that there are two sequences

$$\{a_n\}_{n=1}^{\infty},\ \{b_n\}_{n=1}^{\infty},\ a_n,b_n\to\infty$$

such that

$$\begin{aligned} & S_1(a_n)>A(\ln a_n)^{1/3}(\ln\ln a_n)^{-10/3}, \\ & S_1(b_n)<-B(\ln a_n)^{1/3}(\ln\ln a_n)^{-10/3}; \\ & A,B>0. \end{aligned} \tag{3.1}$$

Since

$$S_1(t),\ t>0$$

is the continuous function then by (3.1) there is (Bolzano-Cauchy) the sequence

$$\{\mu_n\}_{n=1}^{\infty}:\ S_1(\mu_n)=0,\ \mu_n\to\infty, \tag{3.2}$$

where μ_n is the odd-order root of the equation

$$S_1(t)=0,\ t>0. \tag{3.3}$$

Remark 5. We may suppose, of course, that the sequence (3.2) is *complete* one in the usual sense, the interval

$$(\mu_n,\mu_{n+1})$$

does not contain any other odd-order root of the equation (3.3).

Remark 6. There is no need to discuss (for our purpose) the question about even-order roots of the equation (3.3).

Hence, we have: if

$$\bar{k}=\bar{k}[\alpha_0(T)]:\ \mu_{\bar{k}}<\alpha_0(T)<\mu_{\bar{k}+1} \tag{3.4}$$

and (of course, see (2.4), (3.3))

$$S_1[\alpha_0(T)]\neq 0,$$

then

$$S_1[\alpha_0(T)]=\int_0^{\alpha_0(T)}S(t)\mathrm{d}t=\int_0^{\mu_{\bar{k}}}+\int_{\mu_{\bar{k}}}^{\alpha_0(T)}=\int_{\mu_{\bar{k}}}^{\alpha_0(T)}S(t)\mathrm{d}t. \tag{3.5}$$

Consequently, we have from (2.4) by (3.4), (3.5) the following

Corollary 2.

$$\begin{aligned} & \left|\int_{\mu_{\bar{k}}}^{\alpha_0(T)}\arg\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t\right|\sim \\ & \sim\pi(c_l)^{\frac{1}{2l}}\prod_{r=1}^{k}\left|\frac{\zeta\left(\frac{1}{2}+i\alpha_r(T,l)\right)}{\zeta\left(\frac{1}{2}+i\beta_r(T)\right)}\right|^{-\frac{1}{l}},\ T\to\infty. \end{aligned} \tag{3.6}$$

3.2. Next, we obtain from (3.6), (comp. (2.8)), the following

Corollary 3.

$$
\begin{split}
& \left| \left.\langle \arg\zeta\left(\frac{1}{2}+it\right)\rangle\right|_{[\mu_{\bar{k}},\alpha_0(T)]}\right| \sim \\
& \sim \frac{\pi(c_l)^{\frac{1}{2l}}}{\alpha_0(T)-\mu_{\bar{k}}}\prod_{r=1}^k \left|\frac{\zeta\left(\frac{1}{2}+i\alpha_r(T,l)\right)}{\zeta\left(\frac{1}{2}+i\beta_r(T)\right)}\right|^{-\frac{1}{l}},
\end{split} \tag{3.7}
$$

and, of course, (see (2.8), (3.7))

$$
\begin{split}
& \left| \left.\langle \arg\zeta\left(\frac{1}{2}+it\right)\rangle\right|_{[\mu_{\bar{k}},\alpha_0(T)]}\right| \sim \\
& \sim \frac{\alpha_0(T)}{\alpha_0(T)-\mu_{\bar{k}}}\left| \left.\langle \arg\zeta\left(\frac{1}{2}+it\right)\rangle\right|_{[0,\alpha_0(T)]}\right|, \ T\to\infty.
\end{split}
$$

4. On infinite set of metamorphoses of the Q-system that is generated by the factorization formula (2.4)

4.1. Let us remind the Riemann-Siegel formula

$$
Z(t)=2\sum_{n\leq\tau(t)}\frac{1}{\sqrt{n}}\cos\{\vartheta(t)-t\ln n\}+\mathcal{O}(t^{-1/4}), \tag{4.1}
$$

where

$$
\begin{split}
& Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right), \ \tau(t)=\sqrt{\frac{t}{2\pi}}, \\
& \vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right),
\end{split}
$$

(see [5], pp. 79, 239). Next, we have introduced (see [2], (2.1)) the following oscillatory Q-system (based exactly on the Riemann-Siegel formula (4.1))

$$
\begin{split}
& G(x_1,\dots,x_k;y_1,\dots,y_k)=\prod_{r=1}^k\left|\frac{Z(x_r)}{Z(y_r)}\right|= \\
& =\prod_{r=1}^k\left|\frac{\sum_{n\leq\tau(x_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(x_r)-x_r\ln n\}+R(x_r)}{\sum_{n\leq\tau(y_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(y_r)-y_r\ln n\}+R(y_r)}\right|, \\
& (x_1,\dots,x_k)\in M_k^1, \ (y_1,\dots,y_k)\in M_k^2, \\
& R(t)=\mathcal{O}(t^{-1/4}), \ k\leq k_0\in\mathbb{N},
\end{split} \tag{4.2}
$$

where

$$
\begin{split}
& M_k^1=\{(x_1,\dots,x_k)\in(T_0,+\infty)^k, \ T_0<x_1<\dots<x_k\}, \\
& M_k^2=\{(y_1,\dots,y_k)\in(T_0,+\infty)^k, \ T_0<y_1<\dots<y_k\}, \\
& x_r,y_r\neq\gamma: \ \zeta\left(\frac{1}{2}+i\gamma\right)=0, \ r=1,\dots,k.
\end{split} \tag{4.3}
$$

4.2. Next, we have obtained (see [2], (3.1)) the following spectral formula

$$\begin{split} & Z(t)=2\sum_{n\leq\tau(x_r)}\frac{1}{\sqrt{n}}\cos\left\{t\ln\frac{\tau(x_r)}{n}-\frac{x_r}{2}-\frac{\pi}{8}\right\}+ \\ & +\mathcal{O}(x_r^{-1/4}),\ \tau(x_r)=\sqrt{\frac{x_r}{2\pi}}, \\ & t\in[x_r,x_r+V],\ V\in(0,\sqrt[4]{x_r}], \end{split} \tag{4.4}$$

(and similarly for $x_r\longrightarrow y_r$), where

$$T_0<x_r,y_r,\ r=1,\dots,k.$$

Remark 7. The spectral formula (4.4) is, of course, a variant of the Riemann-Siegel formula (4.1).

Remark 8. We call the expressions

$$\frac{2}{\sqrt{n}}\cos\left\{t\omega_n(x_r)-\frac{x_r}{2}-\frac{\pi}{8}\right\},\dots \tag{4.5}$$

as the local Riemann's oscillators with:

(a) the amplitudes

$$\frac{2}{\sqrt{n}},$$

(b) the incohorent local phase constants

$$\left\{-\frac{x_r}{2}-\frac{\pi}{8}\right\},\ \left\{-\frac{y_r}{2}-\frac{\pi}{8}\right\},$$

(c) the non-synchronized local times

$$t=t(x_r)\in[x_r,x_r+V],\dots$$

(d) the local spectrum of the cyclic frequencies

$$\begin{split} & \{\omega_n(x_r)\}_{n\leq\tau(x_r)},\ \omega_n(x_r)=\ln\frac{\tau(x_r)}{n}, \\ & \{\omega_n(y_r)\}_{n\leq\tau(y_r)},\ \omega_n(y_r)=\ln\frac{\tau(y_r)}{n}. \end{split}$$

Remark 9. The Q-system (4.2) represents a complicated oscillating process generated by oscillations of big number of the local Riemann's oscillators (4.5).

4.3. Now, in connection with the oscillating Q-system (4.2), the following corollary follows from our Theorem

Corollary 4.

$$\begin{split} & \prod_{r=1}^k\left|\frac{\sum_{n\leq\tau(\alpha_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\alpha_r)-\alpha_r\ln n\}+R(\alpha_r)}{\sum_{n\leq\tau(\beta_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\beta_r)-\beta_r\ln n\}+R(\beta_r)}\right|\sim \\ & \sim\pi^l\sqrt{c_l}\left|\int_0^{\alpha_0(T)}\arg\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t\right|^{-l},\ T\to\infty. \end{split} \tag{4.6}$$

Remark 10. Hence, we have two resp. one parametric sets of control functions (=Golem's shem) for admissible and fixed ϵ, k, (see (2.3)),

$$\begin{aligned} & \{\alpha_0(T,l),\alpha_1(T,l),\dots,\alpha_k(T,l)\}, \\ & \{\beta_1(T),\dots,\beta_k(T)\}, \\ & T\in(T_0,+\infty),\ l\in\mathbb{N}, \end{aligned} \tag{4.7}$$

of the metamorphoses (4.6), (comp. [1],[2]).

Remark 11. The mechanism of the metamorphosis is as follows. Let (comp. (4.3), (4.7))

$$\begin{aligned} & M_k^3=\{\alpha_1(T,l),\dots,\alpha_k(T,l)\}, \\ & M_k^4=\{\beta_1(T),\dots,\beta_k(T)\}, \end{aligned} \tag{4.8}$$

where, of course,

$$\begin{aligned} & M_k^3\subset M_k^1\subset (T_0,+\infty)^k, \\ & M_k^4\subset M_k^2\subset (T_0,+\infty)^k. \end{aligned} \tag{4.9}$$

Now, if we obtain after random sampling of the points

$$(x_1,\dots,x_k),\ (y_1,\dots,y_k)$$

(see the conditions (4.3)) such that

$$\begin{aligned} & (x_1,\dots,x_k)=(\alpha_1(T,l),\dots,\alpha_k(T,l))\in M_k^3, \\ & (y_1,\dots,y_k)=(\beta_1(T),\dots,\beta_k(T))\in M_k^4, \end{aligned} \tag{4.10}$$

(see (4.8), (4.9)), then - at the points (4.10) - the Q-system (4.2) changes its old form (=chrysalis) to the new one (=butterfly), and the last ist controlled by the function $\alpha_0(T)$.

4.4. Now, we rewrite the formula (4.6), (comp. (3.6)), as follows:

$$\begin{aligned} & \left|\int_{\mu_{\bar{k}}}^{\alpha_0(T)}\arg\zeta\left(\frac12+it\right)\mathrm{d}t\right|\sim \\ & \sim \pi(c_l)^{\frac{1}{2l}}\prod_{r=1}^k\left|\frac{\sum_{n\le\tau(\alpha_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\alpha_r)-\alpha_r\ln n\}+R(\alpha_r)}{\sum_{n\le\tau(\beta_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\beta_r)-\beta_r\ln n\}+R(\beta_r)}\right|^{-\frac{1}{l}}. \end{aligned} \tag{4.11}$$

Remark 12. The formula (4.11) expresses the metamorphosis in the reverse direction. We describe the mechanism of this as follows: we begin with the integral

$$\left|\int_0^w\arg\zeta\left(\frac12+it\right)\mathrm{d}t\right|$$

that is the Aaron staff,

$$\longrightarrow\left|\int_{\mu_{\bar{k}}}^{\alpha_0(T)}\arg\zeta\left(\frac12+it\right)\mathrm{d}t\right|$$

that is the bud of the Aaron staff corresponding to $w=\alpha_0(T)$,

$$\sim \pi(c_l)^{\frac{1}{2l}}\prod_{r=1}^k\left|\frac{\sum_{n\le\tau(\alpha_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\alpha_r)-\alpha_r\ln n\}+R(\alpha_r)}{\sum_{n\le\tau(\beta_r)}\frac{2}{\sqrt{n}}\cos\{\vartheta(\beta_r)-\beta_r\ln n\}+R(\beta_r)}\right|^{-\frac{1}{l}}$$

already metamorphosed one into almonds ripened, (motivation: Chumash, Bamidbar, 17:23).

I would like to thank Michal Demetrian for his help with electronic version of this paper.

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Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynska Dolina M105, 842 48 Bratislava, SLOVAKIA

E-mail address: jan.mozer@fmph.uniba.sk